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Analysis of the Stern-Gerlach Measurement *

A. Venugopalan, Deepak Kumar and R. Ghosh

School of Physical Sciences, Jawaharlal Nehru University, New Delhi - 110067, INDIA

A dynamical model for the collapse of the wave function in a quantum measurement process is proposed by considering the interaction of a quantum system (spin-1/2) with a macroscopic quantum apparatus interacting with an environment in a dissipative manner. The dissipative interaction leads to decoherence in the superposition states of the apparatus, making its behaviour classical in the sense that the density matrix becomes diagonal with time. Since the apparatus is also interacting with the system, the probabilities of the diagonal density matrix are determined by the state vector of the system. We consider a Stern-Gerlach type model, where a spin- 1/2 particle is in an inhomogeneous magnetic field, the whole set up being in contact with a large environment. Here we find that the density matrix of the combined system and apparatus becomes diagonal and the momentum of the particle becomes correlated with a spin operator, selected by the choice of the system-apparatus interaction. This allows for a measurement of spin via a momentum measurement on the particle with associated probabilities in accordance with quantum principles.

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I. INTRODUCTION

After about eighty years of remarkable success of quantum mechanics, the understanding of its measurement aspects remains poor and debatable. The root of the difficulty is that while the state vector of a quantum system has a deterministic evolution, the measurable properties of the system can only be predicted probabilistically. The linearity of quantum equations allows for solutions which are a linear superposition of some basis states, each of which may correspond to one of the different values of a dynamical variable. So the definitive outcome that is obtained in a given measurement of this variable can only be accounted for by the notion of the 'collapse' of the wavefunction to one of the basis states contained in the wavefunction. Though the question as to which state the wavefunction collapses to can be answered probabilistically, quantum mechanics contains no mechanism for the collapse. Bohr [1] and von Neumann [2] postulated that this process occurs when the quantum system comes into contact with an apparatus which must be described classically. Moreover, since the state vector can be expressed as a linear superposition of basis states in any number of ways, the interaction between the classical apparatus and the system decides which property would be measured. According to Bohm [3] this implies that the classical properties as we observe them are contained only as potentialities in the state vector. Many authors have expressed dissatisfaction with this dichotomy between the quantum world and the classical world. Are there really two kinds of systems? If so, is there a way to describe the combined dynamics of quantum and classical systems to exhibit the so called 'collapse'? No satisfactory answer to these issues has emerged even after intensive efforts by a large number of workers [4–9].

An interesting line of investigation to resolve this issue was initiated by Zeh [7], who observed that the measurement apparatus, being always a macroscopic object, has closely spaced energy levels which make it very susceptible to the influence of the environment. The environment consists of a large number of degrees of freedom and its interaction with the apparatus causes decoherence in the quantum evolution of the latter. This decoherence is quite a general feature whenever one considers the interaction between a small quantum system and a large one and monitors the density matrix of the small system only [10]. Here the off-diagonal elements of this reduced density matrix decay in time. However such decays seem dissipative only for small times as they are actually arising due to the superposition of a large number of harmonic terms, viz.$\sum_{j=1}^{n} a_j e^{i\omega_j t}$, which, under the condition that ω_j are closely spaced, give rise to apparent decays for time $t \ll T$, where T is a recurrence time which can be astronomically large, even under mild conditions of $n \sim 1000$ and $\Delta\omega \sim 10^{-5}$. Thus a quantum system coupled to an environment consisting of a large number of degrees of freedom behaves like a classical system in the sense that at time scales of interest, its density matrix is driven diagonal. Since the diagonal density matrix is interpretable in classical terms, several authors have examined the effect of environmental interactions and state reduction in a variety of systems and circumstances [7–18].

In this paper our goal is to further this line of inquiry by examining the process of measurement in the following way. We consider a canonical spin-1/2 quantum system interacting with an apparatus, which in turn is interacting with an environment. The basic idea of the scheme is to study the following two essential features of the quantum measurement process. The first is the establishment of correlations between states of the system and the apparatus. The second is the reduction of the density matrix of the macroscopic apparatus to a diagonal form dictated by the correlations of the apparatus with the system. The

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second aspect is achieved here by interaction of the apparatus with the environment. Thus through this scheme one is able to study the interaction of a quantum system with a macroscopic apparatus within the pure realm of quantum mechanics and provide a realization of the early ideas of Bohr.

We consider the set up of a Stern-Gerlach apparatus, i.e., a spin-1/2 particle in the presence of an inhomogeneous magnetic field. A measurement of spin is made by studying the position or momentum of the particle. Such a model has been considered in some detail by Bohm [3]. Here spin plays the role of the system, and the position/momentum degrees of freedom of the particle that of the apparatus. The particle is further coupled to an environment via its position, and this coupling is intended to drive its translational behaviour classical to perform a measurement. The effect of the environment on the translational degrees of freedom is taken into account via the density matrix equation which incorporates both the quantum evolution and the stochastic Fokker-Planck type evolution arising due to the environmental interaction. This equation has been obtained in a variety of ways in recent literature [10,13,15]. We solve this equation exactly by including an appropriate system-apparatus interaction and show that in this case the reduced density matrix is driven diagonal at long times in the momentum space and has the desired correlation with a component of spin.

The remaining paper is organized as follows. Section II contains the calculations and analysis of the density matrix for the Stern-Gerlach apparatus, and section III summarizes our various observations arising from these calculations.

II. DENSITY MATRIX FOR THE STERN-GERLACH APPARATUS

We consider a Stern-Gerlach type set-up for investigating the measurement of spin. The Hamiltonian of the combined system/apparatus and environment is [19]

$$H^{SAE} = p^2/2m + \lambda\sigma_z + \epsilon x\sigma_z + H^{AE} + H^E. \qquad (1)$$

Here x and p denote the position and momentum (taken in one dimension for convenience) of the particle of mass m, $\lambda\sigma_z$ the Hamiltonian of the system, ϵ the product of the field gradient and the magnetic moment of the particle, H^{AE} the interaction of the environmental degrees of freedom with x, and H^E denotes the Hamiltonian for the environmental degrees of freedom. Since the problem of motion of a spinless particle in simple potentials and in interaction with the environment has been studied at great length in recent literature [10,13–15], we draw upon this work to deal directly with a reduced density matrix equation for the particle in which environmental degrees of freedom have been traced over. Though the density matrix equation has been derived in a number of ways, the derivation which is in the spirit of the present work was given by Caldeira and Leggett using Feynman-Vernon path integral approach [10]. In this method the path integral expression for the density matrix is written for the Hamiltonian of (1), and then the degrees of the environment which consists of oscillators are integrated out. In the limit of high temperature (weak coupling) the expression for the reduced density matrix is seen to be a solution of the above mentioned density matrix equation. This density matrix equation can be thought of as a Markovian limit of a Generalized Master equation, and thus its validity lies in the large time domain, $t \gg t_c$, where t_c is a short relaxation time associated with the environment. This limit is needed here as the earlier work [15] shows that the solution then evolves to a classical stochastic distribution. The tracing over environmental degrees of freedom may be construed as if one is not dealing with one measurement process. However, we feel that this is an essential characteristic of the interaction of a system with a macroscopic object and may be viewed in the same spirit as a single measurement is regarded as an ensemble average for macroscopic systems.

We look at the time evolution of the density matrix in the $|s, x\rangle$ representation, where $|s\rangle$ refers to the eigenstates of σ_z and $|x\rangle$ are the position states. Corresponding to the four elements of the spin space $(\uparrow\uparrow, \downarrow\downarrow, \uparrow\downarrow, \downarrow\uparrow)$, the equations for the elements of the reduced density matrix $\rho_{ss'}(x, y, t)$ for our Hamiltonian are:

$$\begin{aligned}\frac{\partial\rho_{ss'}(x,y,t)}{\partial t} = \Bigg[&\frac{-\hbar}{2im}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\\ &-\gamma(x-y)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) - \frac{D}{4\hbar^2}(x-y)^2\\ &+\frac{i\epsilon(xs - ys')}{\hbar} + \frac{i\lambda(s-s')}{\hbar}\Bigg]\rho_{ss'}(x,y,t),\end{aligned} \qquad (2)$$

where $s, s' = +1$ (for $\uparrow$) or -1 (for $\downarrow$). Here γ is the Langevin friction coefficient and D has the usual interpretation of the diffusion coefficient. In the case of a heat bath of harmonic oscillators at temperature T, $D = 2\gamma m k_B T$. It is more convenient to work with variables: $r \equiv (x-y)$, $R \equiv (x+y)/2$. Then the spin diagonal density matrix ρ_d, and the spin off-diagonal density matrix ρ_{od} obey the equations:

$$\frac{\partial\rho_d(R,r,t)}{\partial t} = \frac{-\hbar}{im}\frac{\partial^2\rho_d}{\partial R\partial r} - \gamma r\frac{\partial\rho_d}{\partial r} - Dr^2 4\hbar^2\rho_d \pm \frac{i\epsilon r}{\hbar}\rho_d, \qquad (3)$$

where the '+' sign in the last term corresponds to $\rho_{\uparrow\uparrow}$ and '-' to $\rho_{\downarrow\downarrow}$, and

$$\begin{aligned}\frac{\partial\rho_{od}(R,r,t)}{\partial t} = &\frac{-\hbar}{im}\frac{\partial^2\rho_{od}}{\partial R\partial r} - \gamma r\frac{\partial\rho_{od}}{\partial r} - Dr^2 4\hbar^2\rho_{od}\\ &\pm\frac{2i\epsilon r}{\hbar}\rho_{od} \pm \frac{2i\lambda r}{\hbar}\rho_{od},\end{aligned} \qquad (4)$$

where the upper signs in the last two terms correspond to $\rho_{\uparrow\downarrow}$ and the lower ones to $\rho_{\downarrow\uparrow}$. To solve these equations, it is convenient to take a partial Fourier transform in the variable R:

$$\rho(Q,r,t) = \int_{-\infty}^{\infty} \exp(iQR)\rho(R,r,t)dR. \qquad (5)$$

The equations (3) and (4) simplify to a pair of first-order partial differential equations:

$$\frac{\partial \rho_d(Q,r,t)}{\partial t} = \left(\frac{\hbar Q}{m} - \gamma r\right)\frac{\partial \rho_d}{\partial r} - \frac{Dr^2}{4\hbar^2}\rho_d \pm \frac{i\epsilon r}{\hbar}\rho_d, \quad (6)$$

$$\frac{\partial \rho_{od}(Q,r,t)}{\partial t} = \left(\frac{\hbar Q}{m} - \gamma r\right)\frac{\partial \rho_{od}}{\partial r} - \frac{Dr^2}{4\hbar^2}\rho_{od} \pm \frac{2\epsilon}{\hbar}\frac{\partial \rho_{od}}{\partial Q} \pm \frac{i\lambda r}{\hbar}\rho_{od}, \qquad (7)$$

Such equations, being of first order, can be solved by the method of characteristics [20]. The physical significance of the solution can be clearly understood if we choose the initial condition to be the following Gaussian wave packet of width σ and mean momentum $\overline{p}$:

$$\psi(x,0) = \frac{1}{(\sigma\sqrt{\pi})^{1/2}}\exp(i\overline{p}x - x^2/2\sigma^2). \qquad (8)$$

The solution for (7), i.e., the spin *off-diaginal* elements of the density matrix, for the initial conditions of (8) is (see appendix):

$$\begin{aligned}\rho_{od}(Q,r,t) = {}& \exp\left(\frac{\tau^3\epsilon^2 D}{3m^2\gamma^5\hbar^2}\right)\exp\left(\frac{\pm 2i\lambda t}{\hbar}\right) \\ & \exp\Big[-\frac{1}{4}\left(\frac{D}{\hbar^2\gamma}(1-e^{-2\tau}) + \frac{1}{\sigma^2}e^{-2\tau}\right)r^2 \\ & +\Big[i\overline{p}e^{-\tau} \mp \frac{\epsilon\tau e^{-2\tau}}{\gamma^2 m\sigma^2} \mp \frac{\epsilon D}{2\hbar^2 m\gamma^3}(\tau(1-e^{-2\tau}) \\ & -2(1-e^{-\tau})) \pm \frac{D\epsilon\tau}{\hbar^2 m\gamma}\Big]r - \Big(\frac{1}{4\sigma^2}(1-e^{-\tau})^2 \\ & +\frac{D}{8\hbar^2\gamma}(4\tau - 3 + 4e^{-\tau} - e^{-2\tau})\Big)r_Q^2 \\ & +\Big(ip(1-e^{-\tau}) - \frac{1}{4\sigma^2}(2r \pm \frac{4\epsilon\tau}{\gamma^2 m})(1-e^{-\tau}) \\ & -\frac{Dr}{4\hbar^2\gamma}(1-e^{-\tau})^2 \pm \frac{\epsilon D}{2\hbar^2 m\gamma^3}(1-e^{-\tau})(\tau(1 \\ & -e^{-\tau}) - 2) \pm \frac{D\epsilon\tau 2}{2\hbar^2 m\gamma^2} \mp \frac{D\epsilon\tau}{\hbar^2 m\gamma^2}\Big)r_Q\Big] \\ & \times\exp\left(-\frac{Q^2\sigma^2}{4} - \left(\frac{\epsilon\tau\sigma}{\hbar\gamma}\right)^2\right), \end{aligned} \qquad (9)$$

where $\tau = \gamma t$, and

$$r_Q = Q\hbar/m\gamma \pm 2\epsilon\tau/m\gamma^2 \pm 2\epsilon/m\gamma^2. \qquad (10)$$

The solution has a factor going as $e^{-A\tau^3}$ which drives the entire expression to zero with time, independent of all other arguments in the density matrix. This means that the density matrix is driven diagonal in the spin-space. The time scale over which this happens is given by

$$\tau_s = \left(\frac{3m^2\gamma^5\hbar^2}{\epsilon^2 D}\right)^{1/3}. \qquad (11)$$

The the solution of (6), i.e, for the *spin-diagonal* is (see appendix for details)

$$\begin{aligned}\rho_d(Q,r,t) = \exp\Big[& i\overline{r}_Q - r_Q^2/4\sigma^2 - Q^2\sigma^2/4 \\ & +i\overline{p}(r - r_Q e^{-\tau} - \frac{1}{4}\sigma^2\{(r-r_Q)^2 e^{-2\tau} + 2r_Q \\ & (r-r_Q)e^{-\tau}\} - D/(4\hbar^6\gamma)\{r_Q^2\tau + 2r_Q \\ & (r-r_Q)(1-e^{-\tau}) + (r-r_Q)^2(1-e^{-2\tau})/2\} \\ & \mp i\epsilon/\hbar\gamma\{r_Q\tau + (r-r_Q)(1-e^{-\tau})\}\Big], \end{aligned} \qquad (12)$$

where $r_Q = \hbar Q/m\gamma$. To understand the measurement aspect implied by this solution, we consider the solution in the momentum representation, i.e.,

$$\rho_d(\overline{u},\overline{v},t) = \int \rho_d(x,y,t)e^{i(\overline{u}x+\overline{v}y)}dxdy. \qquad (13)$$

This is obtained by taking a Fourier transform with respect to the variable r in (12) and identifying $Q = \overline{u} - \overline{v}$ and $q = (\overline{u} + \overline{v})/2$. This solution is

$$\begin{aligned}\rho_d(Q,q,t) = 2\sqrt{\frac{\pi}{N(\tau)}}\exp\Big[& \frac{-1}{N(\tau)}\Big[q + \overline{p}e^{-\tau} \\ & \mp\frac{\epsilon}{\hbar\gamma}(1-e^{-\tau}) + \frac{i\hbar Q}{2\sigma^2 m\gamma}e^{-\tau}(1-e^{-\tau}) \\ & -\frac{iQD}{4\hbar\gamma^2 m}(1-e^{-\tau})^2\Big]^2 - \Big[\left(\frac{\hbar}{4\sigma m\gamma}\right)^2 \\ & (1-e^{-\tau})^2 + \sigma^2/4 + \frac{D}{2m^2\gamma^3}(2\tau - 3 + 4e^{-\tau} \\ & -e^{-2\tau})\Big]Q^2 + \Big[\frac{i\overline{p}\hbar}{m\gamma}(1-e^{-\tau}) \mp \frac{i\epsilon\tau}{m\gamma^2} \\ & \pm\frac{i\epsilon}{m\gamma^2}(1-e^{-\tau})\Big]Q\Big], \end{aligned} \qquad (14)$$

where

$$N(\tau) \equiv (D/2\hbar^2\gamma)(1-e^{-2\tau}) + (1/\sigma^2)e^{-2\tau}. \qquad (15)$$

Now we see that the momentum off-diagonal components ($Q \neq 0$) vanish with time, reducing ρ_d to the diagonal form [10,13–15]. The time scale over which this happens is given by

$$\tau_d^{-1} = \frac{DQ^2}{m^2\gamma^3}. \qquad (16)$$

The momentum distribution function can be obtained by looking at the diagonal elements of the density matrix (14), i.e., for $Q = 0$ and $q = \overline{u}$:

$$\rho_d(0, \overline{u}, t) \equiv |\psi(\overline{u})|^2 = 2\sqrt{\frac{\pi}{N(\tau)}} \exp\Big(\frac{-1}{N(\tau)} \{\overline{u} + \overline{p}e^{-\tau} \mp \frac{\epsilon}{\hbar\gamma}(1 - e^{-\tau})\}^2 \Big). \tag{17}$$

This has the classical Ornstein-Uhlenbeck form with the spin-dependent drift caused by the field. In the large t limit, the momentum distribution is centered around $\epsilon/\hbar\gamma$ ($-\epsilon/\hbar\gamma$) for up (down) spin. Thus we see that the measurement of momentum of the particle can determine the spin. It is also interesting to consider this solution in the space coordinates. The diagonal distribution in momentum necessarily implies that the density matrix does not reduce to a diagonal form with respect to space coordinates. The Fourier transform in Q and q of $\rho_d(Q, q, t)$ gives the density matrix in position representation:

$$\begin{aligned}\rho_d(R, r, t) = 2\sqrt{\frac{\pi}{M(\tau)}} \exp\Big[&-\Big[\frac{1}{4\sigma^2}e^{-2\tau} + \frac{D}{8\hbar^2\gamma} (1 - e^{-2\tau})\Big] r^2 + \Big[i\overline{p}e^{-\tau} \mp \frac{i\epsilon}{\hbar\gamma}(1 - e^{-2\tau})\Big] r \\ &- \frac{1}{M(\tau)}\Big[R - \frac{\overline{p}\hbar}{m\gamma}(1 - e^{-2\tau}) \pm \frac{\epsilon}{m\gamma^2}(1 - e^{-2\tau} - \tau) - \frac{i\hbar r}{2\sigma^2 m\gamma}e^{-2\tau} \\ &(1 - e^{-2\tau}) + \frac{iDr}{4m\gamma^2\hbar}(1 - e^{-2\tau})^2\Big], \end{aligned} \tag{18}$$

where $\tau = \gamma t$ and

$$M(\tau) = \sigma^2 + \frac{\hbar^2}{\sigma^2 m^2\gamma^2}(1 - e^{-\tau})^2 + \frac{D}{2m^2\gamma^3}(2\tau - 3 + 4e^{-\tau} - e^{-2\tau}). \tag{19}$$

As $t \to \infty$, one can see that the off-diagonal elements of the density matrix do not vanish, and the diagonal elements give the position distribution function obtained by setting $r = 0$ and $R = x$:

$$|\psi(x)|^2 = 2\sqrt{\frac{\pi}{M(\tau)}} \exp\left(\frac{-1}{M(\tau)}(x - \frac{\overline{p}\hbar}{m\gamma} \pm \frac{\epsilon(1-\tau)}{m\gamma^2} \right)^2. \tag{20}$$

The results corresponding to $\pm\epsilon$ are for the up and down spins. The centers of the position distribution function shift with time and are clearly different for up and down spins. Though the distribution $\rho_d(R, r, t)$ implies a nonlocality through its dependence on r, we observe that the width w_1 of the distribution in r is considerably smaller than the width w_2 of the distribution in R. In the large τ limit, $w_2/w_1 = \hbar^2 m^2\gamma^4/D^2$. If $\rho_d(R, r, t)$ is coarse-grained over length scales l, such that l is larger than the deBroglie wavelength of the particle γ/ϵ and w_1, but smaller than w_2 , we should have a local distribution in position space. This means $l > Max(\gamma/\epsilon, \gamma/D)$, and $l < D\hbar^2\tau/m^2\gamma^2$, which is surely possible for large enough τ.

Thus if the initial wavefunction of the system-apparatus is a product of the Gaussian wave-packet of (8) and the apparatus state $(a|\uparrow\rangle + b|\downarrow\rangle)$, *i.e.*,

$$\psi(x, 0) = \frac{1}{(\sigma\sqrt{\pi})^{1/2}}[a|\uparrow\rangle + b|\downarrow\rangle] \exp(-i\overline{p}x - x^2/2\sigma^2), \tag{21}$$

in the model without the environment, the time evolution of the density matrix is

$$\begin{aligned}\rho = &|a|^2|\uparrow\rangle\langle\uparrow|\psi_+^*(x, t)\psi_+(y, t) \\ &+|b|^2|\downarrow\rangle\langle\downarrow|\psi_-^*(x, t)\psi_-(y, t) \\ &+ab^*|\uparrow\rangle\langle\downarrow|\psi_+^*(x, t)\psi_-(y, t) \\ &+a^*b|\downarrow\rangle\langle\uparrow|\psi_-^*(x, t)\psi_+(y, t),\end{aligned} \tag{22}$$

where $\psi_\pm(x, t)$ are the wavefunctions of the particle in the potential $\pm\epsilon x$. The environment causes the decay of the off-diagonal elements and the large time limit of the density matrix assumes the form

$$\rho_R = |a|^2|\uparrow\rangle\langle\uparrow|\rho_{\uparrow\uparrow} + |b|^2|\downarrow\rangle\langle\downarrow|\rho_{\downarrow\downarrow}, \tag{23}$$

with $\rho_{\uparrow\uparrow}$ and $\rho_{\downarrow\downarrow}$ being given by (17) in the momentum representation, and by (18) in the coordinate representation. This calculation clearly establishes the measurement of spin via a momentum measurement. The spin diagonal density matrix evolves to a diagonal form in the momentum space, while the spin off-diagonal density matrix goes to zero with time. Further, the probability distributions of up and down spins are also given by the initial amlitudes a and b according to the quantum prescription. Figs. 1(a), (b) and (c) show the real part of the sum of the density matrices, $\rho = Real(\rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow})$, given by (17) in the momentum representation for the diagonal spin elements for three different values of the scaled time $\tau = \gamma t$. As τ increases, the off-diagonal elements clearly decay, leaving a diagonal distribution which is centered around two different mean momenta, corresponding to up and down spins. One can identify these mean momenta as those with which the centers of the wave packets corresponding to up and down spins move.

III. SUMMARY OF RESULTS

In this investigation we have considered through a canonical example two essential aspects of the measurement problem, which are (a) decoherence of the superpositions in the apparatus states so as to allow classical inference, and (b) the definite correlation of the system states with the apparatus states. We find that measurement is achieved if the apparatus is macroscopic enough

to be affected by an environment and furthermore, its relevant degree of freedom has a classical limit in the sense of the Correspondence principle. This is to be contrasted with the case in which the relevant degree of freedom of the apparatus has a discrete spectrum, because in that situation the correlation between the system and the apparatus states is not achieved. To conclude, we have been able to provide a scheme of incorporating a concept like "classical apparatus" in a purely quantum formalism and demonstrate that a suitable quantum apparatus when dissipatively coupled to an appropriate environment does perform a measurement. It is in this sense that we justify the concept of Bohr and von Neumann that a measurement requires the interaction of a quantum system with a classical system.

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APPENDIX A:

Equation (6) is equivalent to the following set of ordinary differnetial equations :

$$\frac{dt}{ds} = 1, \tag{A1}$$

$$\frac{dr}{ds} = \gamma(r - r_Q), \tag{A2}$$

$$\frac{d\rho_d}{ds} = -\rho_d \left(\frac{Dr^2}{4\hbar^2} \mp \frac{i\epsilon r}{\hbar} \right), \tag{A3}$$

with $r_Q = \hbar Q/m\gamma$. The invariants of these orbits with respect to s are easily found to be

$$I_1 = (r - r_Q)e^{-\gamma t}, \tag{A4}$$

and

$$I_2 = \rho_d \exp\left[\frac{D}{4\hbar^2} \left(r_Q^2 t + \frac{2r_Q}{\gamma}(r - r_Q) + \frac{(r - r_Q)^2}{2\gamma} \right) \pm \frac{i\epsilon}{\hbar} \left(r_Q t + \frac{r - r_Q}{\gamma} \right) \right]. \tag{A5}$$

Clearly, $I_2 = w(I_1)$, where w is an arbitrary function. This enables us to write

$$\rho_d(Q, r, t) = w(I_1) \exp\left[\frac{D}{4\hbar^2} \Big(r_Q^2 t + \frac{2r_Q}{\gamma}(r - r_Q) + \frac{(r - r_Q)^2}{2\gamma} \Big) \pm \frac{i\epsilon}{\hbar} \left(r_Q t + \frac{r - r_Q}{\gamma} \right) \right], \tag{A6}$$

$w(I_1)$ is now determined from the initial condition (8) for $\rho_d(Q, r, 0)$. One can easily see that

$$\begin{aligned} w(I_1) &= w\left((r - r_Q)e^{-\gamma t} \right) \\ &= \exp\Big[i\overline{p}(r - r_Q)e^{-\gamma t} \\ &\quad - \frac{1}{4\sigma^2}\{(r - r_Q)^2 e^{-2\gamma t} + 2r_Q(r - r_Q)e^{-\gamma t}\} \\ &\quad + \frac{D}{4\hbar^2}\Big(\frac{2r_Q}{\gamma}(r - r_Q)e^{-\gamma t} \\ &\quad + \frac{(r - r_Q)^2 e^{-2\gamma t}}{2\gamma} \Big) \pm \frac{i\epsilon}{\hbar}(r - r_Q)e^{-\gamma t} \Big]. \end{aligned} \tag{A7}$$

Substituting this in (A6) gives the result (12) for $\rho_d(Q, r, t)$. To solve (7) we first make the transformation

$$\rho_{od} = W \exp(\mp 2i\lambda t). \tag{A8}$$

The equation for $W(Q, r, t)$ is now equivalent to the following set of differential equations :

$$\frac{dt}{ds} = 1, \tag{A9}$$

$$\frac{dr}{ds} = \gamma(r - r_Q), \tag{A10}$$

$$\frac{dQ}{ds} = \pm 2\epsilon/\hbar, \tag{A11}$$

$$\frac{dW}{ds} = \frac{-Dr^2}{4\hbar^2}W. \tag{A12}$$

The invariants for this set of equations are :

$$I_1 = \hbar/m\gamma(Q \pm 2\epsilon t/\hbar) \tag{A13}$$

$$I_2 = (r - \hbar Q/\gamma m \pm 2\epsilon/m\gamma^2)e^{-\gamma t}. \tag{A14}$$

The third invariant from (A11) is obviously a fiction of I_1 and I_2, hence,

$$\begin{aligned} W(Q, r, t) = f(I_1, I_2) \exp\Big[&-\frac{D}{4\hbar^2}\Big\{ \left(I_1 \mp \frac{2\epsilon}{m\gamma^2} \right)^2 t \\ &+ 2\left(I_1 \mp \frac{2\epsilon}{m\gamma^2} \right) \left(\frac{I_2 e^{-\gamma t}}{\gamma} \mp \frac{\epsilon t^2}{m\gamma} \right) + \frac{I_2^2 e^{-2\gamma t}}{2\gamma} \\ &\mp \frac{4I_2\epsilon}{m\gamma^3} e^{-\gamma t}(\gamma t - 1) + \frac{4\epsilon^2 t^3}{3m^2\gamma^2} \Big\} \Big]. \end{aligned} \tag{A15}$$

$f(I_1, I_2)$ can now be easily determined from the initial condition (8) for $\rho(Q, r, 0)$:

$$\begin{aligned} f(I_1, I_2) = \exp\Big[&-\frac{\sigma^2}{4}(Q \pm \frac{2\epsilon t}{\hbar})^2 - \frac{1}{4\sigma^2}\Big\{ \left(\frac{\hbar Q}{m\gamma} \mp \frac{2\epsilon t}{\gamma^2} \right) \\ &(1 - e^{-\gamma t}) + re^{-\gamma t} \pm \frac{2\epsilon t}{\gamma} \Big\}^2 + i\overline{p}\Big\{ \Big(\frac{\hbar Q}{m\gamma} \\ &\mp \frac{2\epsilon t}{\gamma^2} \Big)(1 - e^{-\gamma t}) + re^{-\gamma t} \pm \frac{2\epsilon t}{\gamma} \Big\} \Big] \\ &\exp\Big[\frac{D}{2\hbar^2\gamma} \left(\frac{\hbar Q}{m\gamma} \pm \frac{2\epsilon t}{\gamma^2} \mp \frac{2\epsilon}{m\gamma^2} \right) \Big(r - \frac{\hbar Q}{m\gamma} \\ &\pm \frac{2\epsilon}{m\gamma^2} \Big) e^{-\gamma t} + \frac{D}{2\hbar^2\gamma} \Big(r - \frac{\hbar Q}{m\gamma} \pm \frac{2\epsilon}{m\gamma^2} \Big)^2 e^{-\gamma t} \\ &\pm \frac{D\epsilon}{m\gamma^3} \Big(r - \frac{\hbar Q}{m\gamma} \pm \frac{\epsilon}{m\gamma^2} \Big) e^{-\gamma t} \Big]. \end{aligned} \tag{A16}$$

Substituting for $f(I_1, I_2)$ in (A15) gives the result of (9).

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FIG. 1. Plot of the sum of the real part of the spin-diagonal density matrices in the momentum representation, $\rho = Real(\rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow})/\sigma$, given by Eq.(11), versus dimensionless momenta $u = (q + Q/2)\sigma$ and $v = (q - Q/2)\sigma$, for (a) $\tau \equiv \gamma t = 0$, $\epsilon/m\gamma^2 = 0.0$, (b) $\tau = 1$, $\epsilon/m\gamma^2 = 2.0$, (c) $\tau = 3$, $\epsilon/m\gamma^2 = 2.0$, with $\overline{p} = 0.2/\sigma$, $D/m^2\gamma^3 = \sigma^2$, $m\gamma/\hbar = 0.5/\sigma^2$.

Figure 1(a)

Figure 1(b)

Figure 1(c)